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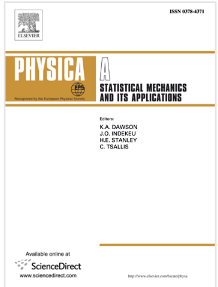

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Dynamics of Two-Actor Cooperation-Competition Conflict Models

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Abstract: We present a nonlinear ordinary differential equation model of the conflict between two actors, who could be individuals, groups, or nations. The state of each actor depends on its own state in isolation, its previous state in time, its inertia to change, and the positive feedback (cooperation) or negative feedback (competition) from the other actor. We analytically determined the stability of the critical points of the model and explored its dynamical behavior through numerical integrations and analytical proofs. Some results of the model are consistent with previously observed characteristics of conflicts and other results make new testable predictions on how the dynamics of a conflict and its outcome depend on the strategies chosen by the actors.

1 Introduction

Our ability to achieve personal happiness and the global stability of our species depends on our ability to understand and deal with conflicts between individuals, groups of people, and nations. Real life conflicts can have many causes that interact with each other and that depend differently on the details of each conflict [1-4]. However, many different conflicts share some common elementary features. One such basic feature is how each actor (which can be a person, group, or nation) responds to another actor. Deutsch [5,6] developed the seminal idea that many different types of conflict reflect the effects of either cooperation or the competition between the actors. "To put it colloquially, if you're positively linked with another, then you sink or swim together; with negative linkage, if the other sinks, you swim, and if the other swims, you sink" [6]. Previous models of conflict have been based on either qualitatively defined reaction functions between the actors [7-9], or a metaphorical application of the concepts from dynamical systems [10-12] or were based on linear mathematical models [13,14] or piecewise linear models [15,16]. A motivating element in these formal mathematical models is that "The mathematics forces the development of precise theory about the mechanisms to create movement in the system. ... We hope to get creative investigators to move away from metaphors of dynamics toward real dynamical equations" [16]. Our goal here is to combine the formal mathematical model of Gottman et al. [15,16] with the intuitive insights of Deutsch [5,6] to form a new mathematical model that will give us insight into how the dynamics of a conflict depends on the cooperation or competition between two actors. The intent is not to make a complete mathematical model of human beings or their behavior. The goal is to use the mathematics as a tool to rigorously determine the logically necessary consequences of the assumptions of the model. We will show that some of the results of this model match previous observations about conflicts and that other results provide interesting and new testable predictions about conflicts.

2 Model

Let x and y represent the emotional state of each actor at time t.

$$\begin{aligned} \frac{dx}{dt} &= m_1 x + b_1 + f_1(y, x) \\ \frac{dy}{dt} &= m_2 y + b_2 + f_2(x, y) \end{aligned} \tag{1}$$

where $m_1 < 0$ and $m_2 < 0$ are the "inertia" terms, b_1, b_2 are the uninfluenced state of each actor alone and $f_1(y, x)$ and $f_2(x, y)$ are influence functions of actor y on actor x and vice versa, respectively. (The parameters m_1, m_2 are time constants of an exponential relaxation for this first order differential equation, rather than a mass-like inertial term present in a second order differential equation. However, as these terms determine the time relaxation of the system, we follow the nomenclature used by Gottman et al. [15,16] and refer to them as the inertial terms.) We represent cooperation as positive feedback between the groups, that is, a positive state of one actor increases the positive state of the other actor and a negative state of one actor increases the negative state of the other actor. Competition is modeled as negative feedback, that is, a positive state of one actor increases the negative state of the other actor and a negative state of one actor increases the positive state of the other group. As shown in Figure 1, we chose a hyperbolic tangent function to represent this feedback. We chose those functions for three reasons: 1) we want each actor to influence the other actor approximately proportionately at small influence levels, 2) in order to prevent the states of each actor from escaping to infinity, we want a feedback function with a plateau at high influence levels, and 3) we want each actor to influence the state of the other actor through either positive or negative feedback that can be switched by switching the sign of a single parameter.

$$\begin{aligned} f_1(y, x) &= c_1 \tanh(y) \\ f_2(x, y) &= c_2 \tanh(x) \end{aligned} \tag{2}$$

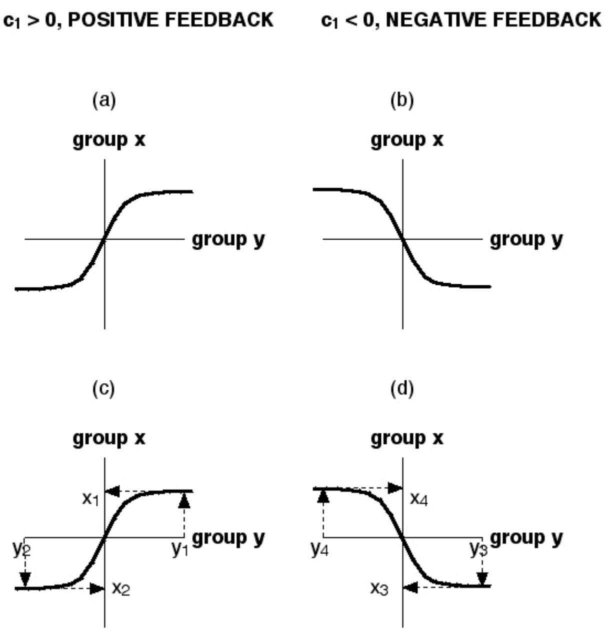

Figure 1: The positive (a, c) and negative (b, d) feedback influence functions from eq. (2). The influence of group y on group x is illustrated. The influence of group x on group y would be similar.

As described in the Appendix, we first compute the nullclines, $dx/dt = 0$ and $dy/dt = 0$, whose intersection determines the critical points which could be stable points where both x and y remain constant in time. The stability of those critical points are then determined from their eigenvalues computed from a linear stability analysis. We also numerically integrate these equations from different initial conditions using ODE113 in Matlab. There are very many combinations of the values of the parameters that we can study. We will concentrate on models with the same inertia to change ($m_1 = m_2 = -0.9$) and the same effect of the uninfluenced sate ($b_1 = b_2 = 0$) with various values of the strength of the feedback between the groups (c_1 and c_2). We are particularly interested in understanding the dynamics of three different cases:

1) where there is positive feedback (cooperation) between both actors,

2) where there is negative feedback (competition) between both actors, and

3) where there is a mixed case in which one actor responds with positive feedback and the other with negative feedback.

We also explore how the actions of one actor alone, by temporarily changing their type of feedback, can alter the dynamics and steady state result of the conflict. The details of the computations are presented in the Appendix.

3 Results

3.1 Weak feedback

The numerical integrations in time for weak feedback are shown in Figure 2. For all three cases, when the strength of the feedback is less than a threshold equal to the inertia to change, then both actors evolve to a neutral state. Interestingly, there is no evidence in the final stable state of the system that there was any feedback between the actors at all, even though there is always a proportional influence between them.

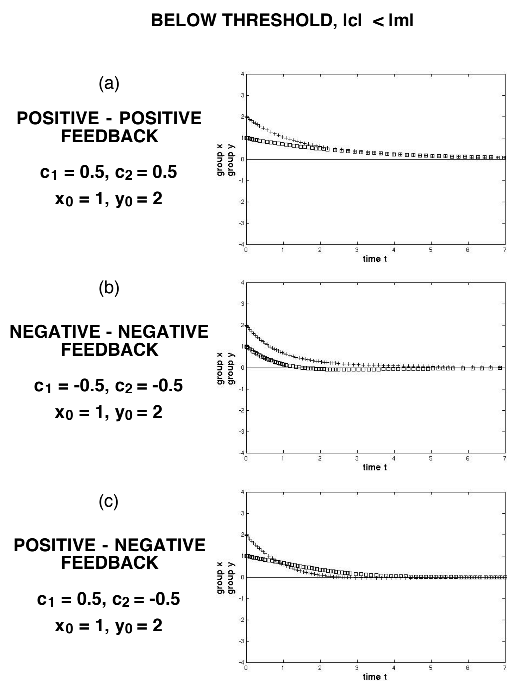

Figure 2: Numerical integration was used to compute the values of x (boxes) and y (plus signs) as the state of the two groups evolves from their initial values of $x = 1$ and $y = 2$. When the strength of the feedback is less than the absolute value of the inertia to change, then all the models evolve to the neutral state $x = 0$ and $y = 0$.

3.2 Strong Positive-Positive Feedback

The nullclines for increasing positive-positive feedback are shown in Figure 3. As the strength of the feedback increases beyond the threshold equal to their inertia, there is a bifurcation in the dynamical behavior. Above this threshold two new critical points appear. The previously stable fixed point at $(x, y) = (0, 0)$ now has one positive and one negative eigenvalue and thus becomes an unstable saddle. The eigenvalues of the two new critical points are negative and are thus stable fixed points. Some illustrative examples of the numerical integrations in time for positive-positive feedback starting at different initial conditions are shown in Figure 4. Depending sensitively on the initial conditions and their uninfluenced states, both actors evolve toward either a positive or a negative state, that is, they either swim or sink together, as described by Deutsch.

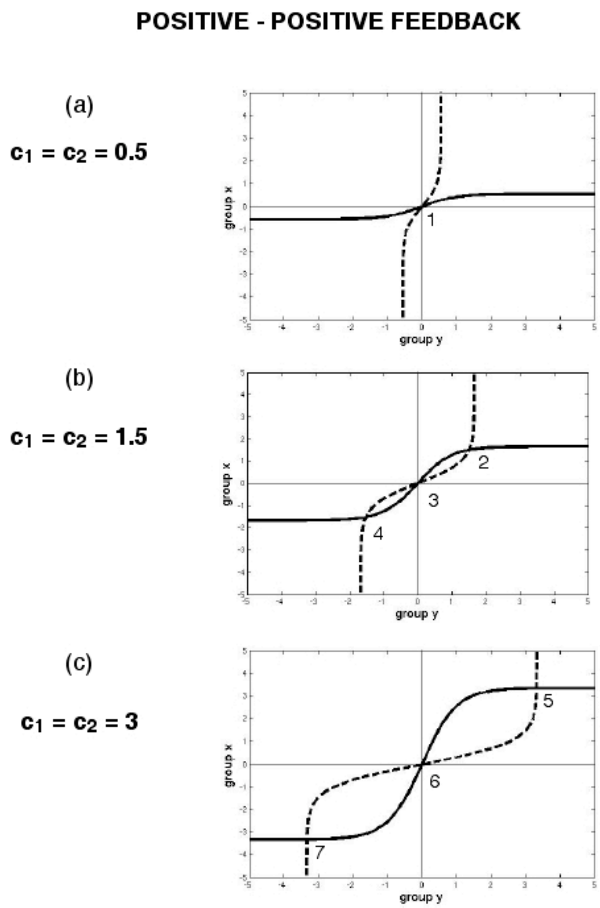

Figure 3: The intersection of the nullclines for $dx/dt = 0$ (solid line) and $dy/dt = 0$ (dashed line) for positive feedback between both groups determine the critical points. The stability of these critical points is evaluated in the Appendix . (a) When the strength of the feedback is weak there is only one fixed point (1). (b,c) As the strength of the feedback increases two new fixed points appear (2,4 and 5, 7) and the original critical point (3, 6) becomes an unstable saddle.

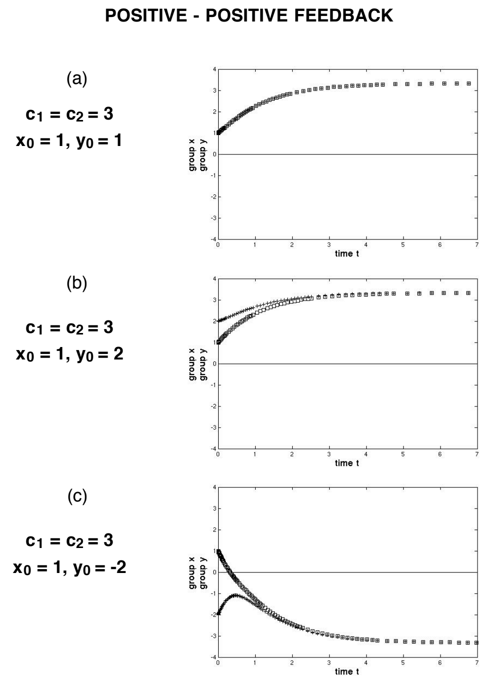

Figure 4: Numerical integration with positive feedback between both groups. The values of x (boxes) and y (plus signs) are shown as the state of the two groups evolves from different initial values of x and y. The values of x and y always evolve toward the stable states where either both are positive or both are negative.

3.3 Strong Negative-Negative Feedback

The nullclines for negative-negative feedback are shown in Figure 5. Similar to the positive-positive feedback case, as the strength of the feedback between the actors increases, the previously stable fixed point a $(x, y) = (0, 0)$ is transformed into an unstable saddle and two new stable fixed points are generated. Numerical integrations in time for two actors with exactly the same parameters and initial conditions are shown in Figure 6. Surprisingly, even though the strength of the feedback is above the threshold equal to the inertia, both actors evolve towards a neutral state at $(x, y) = (0, 0)$. As derived in the Appendix, when the two actors have identical parameters and initial conditions the system is not structurally stable and infinitesimal changes in the parameters or initial conditions cause symmetry breaking in the dynamics. (This is true for only the negative-negative feedback case and not the positive-positive feedback case.) As shown in the numerical integrations in Figure 7, a small 0.1% difference in either the initial conditions (Figure 7b) or parameters (Figure 7c) dramatically changes the dynamics. Both actors first evolve toward a neutral emotional state at the $(x, y) = (0, 0)$ unstable saddle and then separate into opposite positive and negative emotional states. Again, as observed by Deutsch, one swims and other sinks. The approach to and the subsequent separation from the neutral state is an interesting prediction. Visual inspection of data resembling the simulation in Figures 7b and 7c might suggest that a precipitating event occurred at approximately time $t = 3$ to split the actors into diverging trajectories. That is not the case in this model. Here, it is the sensitivity to the initial conditions and parameters at $t = 0$, that established the future direction of the trajectories, both to first approach, and then to deviate from a neutral emotional state.

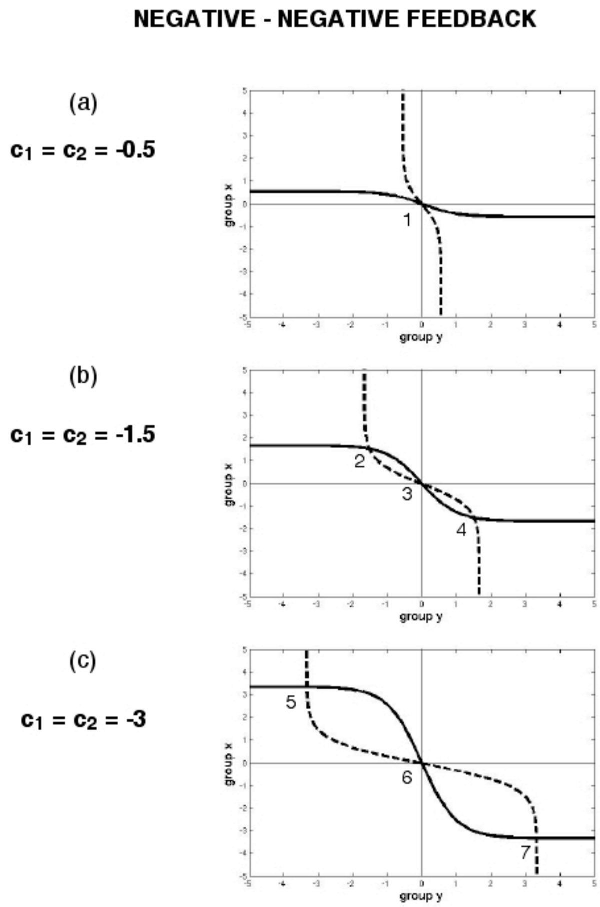

Figure 5: The intersection of the nullclines for $dx/dt = 0$ (solid line) and $dy/dt = 0$ (dashed line) for negative feedback between both groups determine the critical points. The stability of these critical points is evaluated in the Appendix . (a) When the strength of the feedback is weak there is only one fixed point (1). (b,c) As the strength of the feedback increases two new fixed points (2, 4 and 5, 7) appear and the original critical point (3, 6) becomes an unstable saddle.

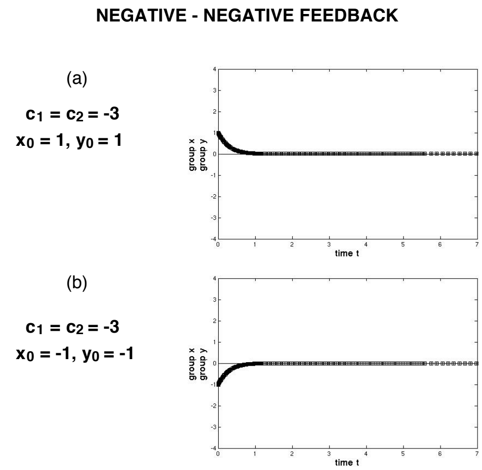

Figure 6: Numerical integration with negative feedback between both groups. The values of x (boxes) and y (plus signs) are shown as the state of the two groups evolves from initial values of x and y. When both groups have the same parameters and the same initial conditions both groups evolve to the neutral state $x = 0$ and $y = 0$.

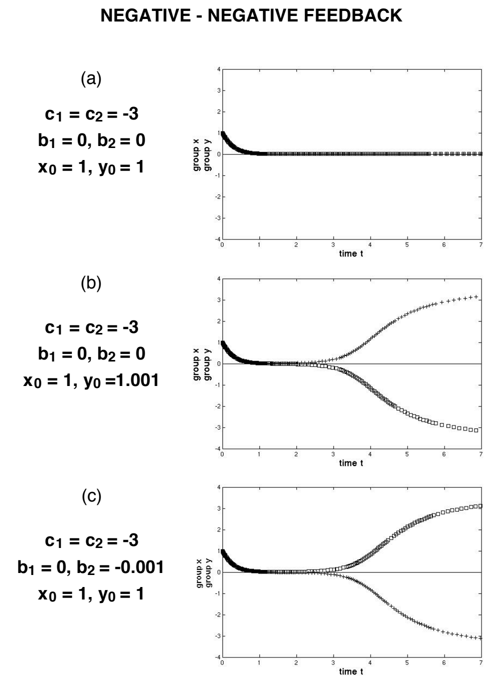

Figure 7: Numerical integration with negative feedback between both groups. The values of x (boxes) and y (plus signs) are shown as the state of the two groups evolves from initial values of x and y. (a) When the groups have the same parameters and initial conditions they both evolve to the neutral state $x = 0$ and $y = 0$. However small differences between either the initial states or the parameters of the groups causes symmetry breaking and the groups evolve towards very different stable states. (b) When the initial state of group x is 0.1% lower than that of group y, group x evolves to a negative state and group y to a positive state. (c) When the value of the effect of the uninfluenced state of group x is 0.1% larger than group y, group x evolves to a positive state and group y to a negative state.

3.4 Strong Positive-Negative Feedback

The nullclines for positive-negative feedback are shown in Figure 8 (The nullclines for negative-positive feedback are the mirror images of the positive-negative feedback nullclines.) In this case, for all strengths of feedback, there is only one critical point, whose eigenvalue is complex with a negative real component, so that $(x, y) = (0, 0)$ is a focus. Thus, as shown by the numerical integrations in Figure 9, the emotional states of both actors oscillate with decaying amplitude as they evolve toward their neutral steady state. This case also illustrates the utility of a rigorously defined mathematical model and its analysis. Although it is not possible to use words to easily reason out the dynamical consequences of this positive-negative feedback case, the mathematical stability and dynamical analysis is straightforward. Moreover, mathematical analysis gives rise to two important new predictions. First, it predicts the existence of these oscillations in emotions. Second, it predicts that an intractable negative-negative feedback conflict with a clear winner and loser can be switched into at least a neutral condition for both actors by the action of one actor alone unilaterally switching their strategy to positive feedback. The full bifurcation structure of all three cases is shown in Figure 10.

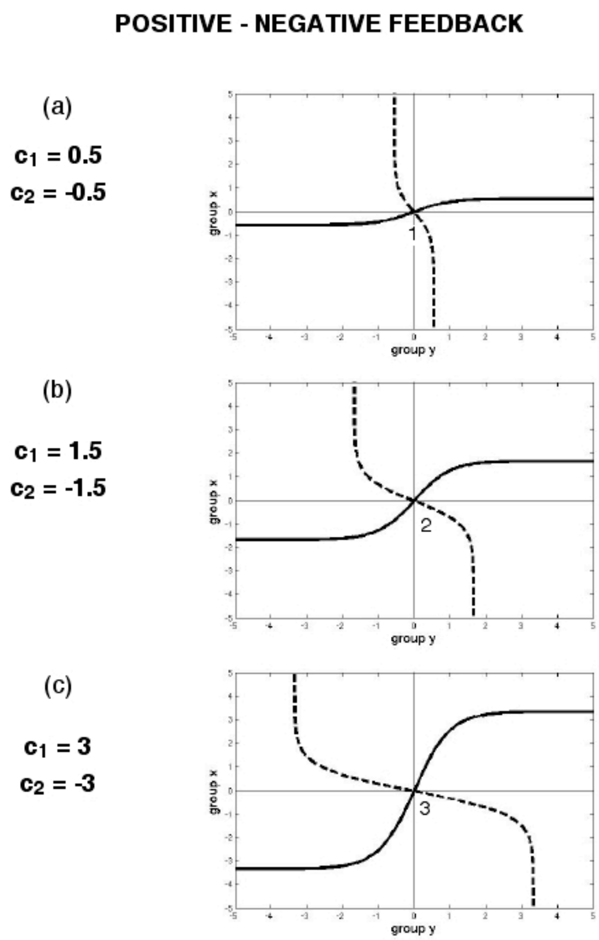

Figure 8: The intersection of the nullclines for $dx/dt = 0$ (solid line) and $dy/dt = 0$ (dashed line) for positive-negative feedback between the groups determine the critical points. The stability of these critical points is evaluated in the Appendix. For all strengths of the feedback there is only critical point (1,2,3) which is stable.

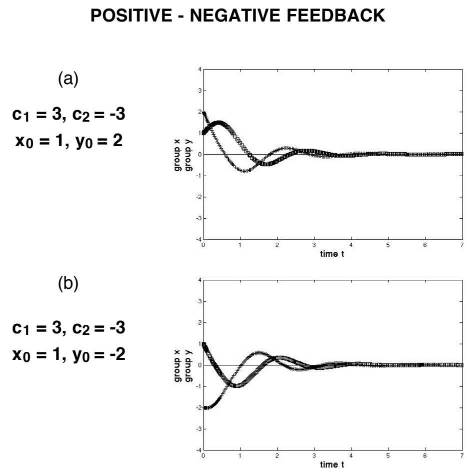

Figure 9: Numerical integration with positive feedback from group y to group x and negative feedback from group x to group y. The values of x (boxes) and y (plus signs) are shown as the states of the two groups evolve from different initial values of x and y. The values of x and y oscillate as they evolve toward the neutral state $x = 0$ and $y = 0$.

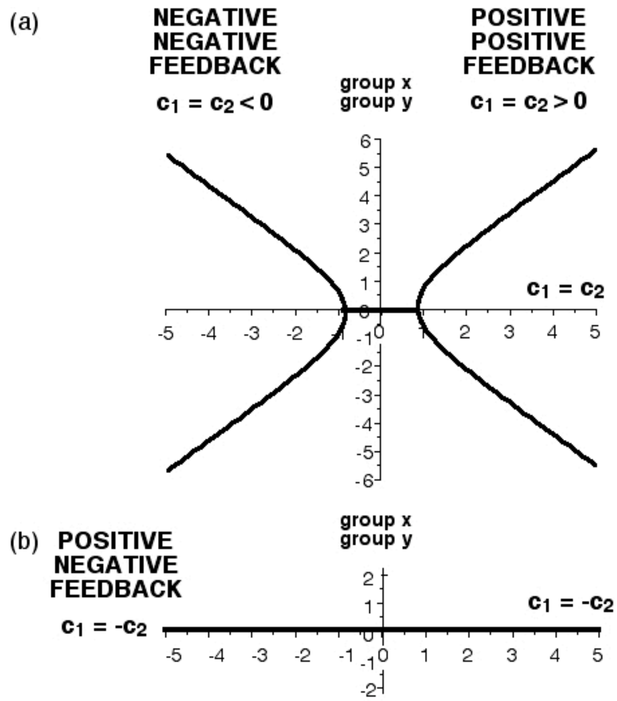

Figure 10: The bifurcation diagram summarizes the dependence of the stable states on the strength of the feedback between the groups. On these plots, the abscissa is the strength of the interaction between the groups c_1 and the ordinate is emotional state of each group given by the values of x or y at their steady state values. (a) When there is positive feedback between both groups (and $c_1 = c_2$) the neutral state $x = 0$ and $y = 0$ is stable until the strength of the feedback exceeds the absolute value of the inertia to change. When the strength of the positive feedback is larger than that threshold, then the neutral state $x = 0$ and $y = 0$ is unstable and there are two stable states with x and y both positive or x and y both negative. When there is negative feedback between both groups (and $c_1 = c_2$) the neutral state $x = 0$ and $y = 0$ is stable until the strength of the feedback exceeds the absolute value of the inertia to change. When the strength of the negative feedback is larger than that threshold, then the neutral state $x = 0$ and $y = 0$ is unstable (if there is any difference in the parameters or the initial states between the groups) and there are two stable states, one with x positive and y negative, and one with x negative and y positive. (b) When there is positive feedback from one group and negative feedback from the other group (and $c_1 = -c_2$), then only the neutral state $x = 0$ and $y = 0$ is stable.

3.5 The Effect of One Actor Alone Temporarily Changing their Feedback

The existence of oscillations in the positive-negative feedback case gives rise to an interesting and testable prediction. Both the positive-positive feedback and negative-negative feedback cases reach fixed points that are either advantageous or disadvantageous to each actor. However, perhaps surprisingly, those stable outcomes can be reversed by one actor alone, unilaterally switching their feedback for a duration of time. If that controlling actor first switches their feedback from positive to negative (in the positive-positive feedback case) or from negative to positive (in the negative-negative feedback case) the positive-negative feedback present will result in an oscillation of the emotional states of both actors. The emotional state of each actor will now oscillate up and down with different phases. If that controlling actor now switches their feedback a second time, to return to their original feedback, the relative emotional states of both actors at that time will depend on the duration of time between the first and second switches in feedback. The emotional state of both actors will now evolve as if the original computation was rerun with the new initial conditions present at the moment of that second switch in feedback. Since the outcome is sensitive to these initial conditions, both actors may now reach new stable fixed points. The duration of time between the first and second switches in feedback determine the relative emotional states of the two actors and thus their final stable states. Hence, by properly choosing this duration, one actor alone can reverse the role of winner or loser in a conflict. This behavior is illustrated in Figure 11 for the negative-negative feedback case. The two actors evolve from time $t = 0$, the first actor reaching the negative fixed point and the second actor the positive fixed point. In Figure 11a, at time $t = 6$, the first actor switches their strategy to positive feedback for a duration $\mathrm{DT} = 0.75$, and then both actors evolve to their previous fixed points. On the other hand, in Figure 11b, if at time $t = 6$, the first actor switches their strategy to positive feedback for a duration $\mathrm{DT} = 1.50$, then the actors evolve to the opposite fixed points, and the role of winner and loser has been reversed. The duration of the time between the first and second switch that will reverse the role of winner and loser is proportional to the period of the oscillation in the positive-negative feedback case, which as shown in the Appendix, is inversely proportional to the square root of the product of the strength of the feedback between the two actors. Moreover, this transient feedback switch would only be useful to repeat on the time scale at which the actors reach their fixed point values which is inversely proportional to the inertia of the actors. Thus, the model predicts that

1) the more the inertia to change, the more often an actor can gain an advantage by switching strategies, and

2) the stronger the feedback between the actors, the shorter should be the duration of their switch to gain this advantage.

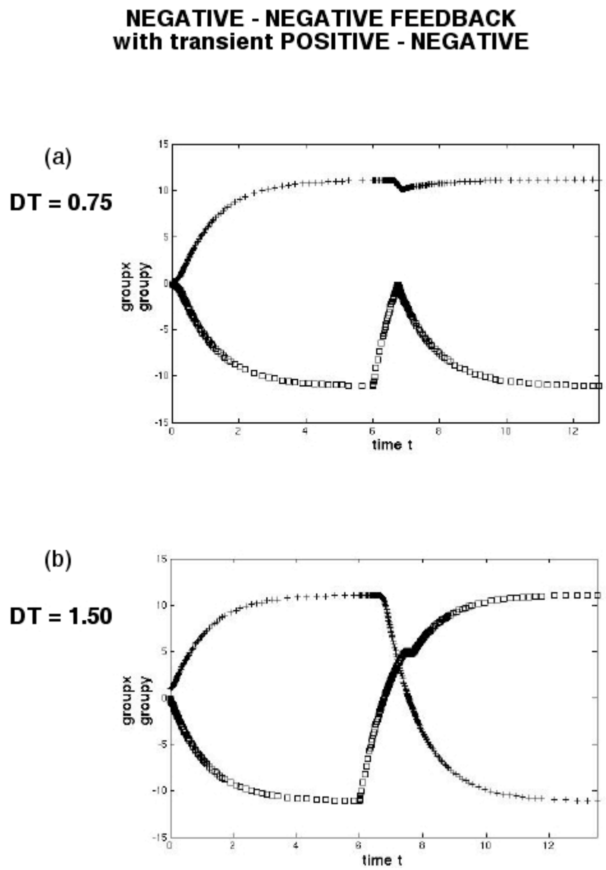

Figure 11: One group, acting unilaterally can reverse the role of winners and losers in negative-negative feedback. Numerical integration of the values of x (boxes) and y (plus signs) as a function of time. (a) At time $t = 6$, group x temporarily switches their strategy to positive feedback for a duration DT = 0.75, and then both groups evolve to their previous fixed points. (b,) At time $t = 6$, group x temporarily switches their strategy to positive feedback for a duration DT = 1.50, and then the actors evolve to the opposite fixed points.

3.6 Two Actors - Global Dynamical Behavior - Analytical Methods to Detect Limit Cycles

The continuous set of differential equations formulated in the model makes it possible to use analytical methods to determine important properties of the dynamics of the model, and how these dynamics depend on the parameters used in eqs. (1). We have already illustrated some dynamical properties by using specific values of the parameters. In each of the cases the trajectories of the dynamics converge to equilibria. We now ask whether there are other sets of parameter values that would display markedly different dynamical behavior, for example, limit cycles. We will now prove that for two actors the dynamics does not contain limit cycles. Thus, this analysis shows that the specific examples that we have already presented for the dynamics of two actors covers all the types of dynamics present in the model. Determining the existence of limit cycles is very challenging. The eigenvalue analysis presented in the Appendix is a linear and local analysis around the critical points which therefore does not provide us information on whether there are global trajectories, that pass through the domains far from those critical points. There is no single inclusive procedure for determining the existence of limit cycles in dynamical systems, this is a non-trivial problem. In order to study the global properties of the dynamics, first, we now prove that for two actors, under a very wide range of parameters, that the dynamics do not contain limit cycles. In a later section we extend these results to certain sets of parameters for systems with three actors.

Again, we consider the case for ($b_1 = b_2 = 0$) and rescale eq. (1) in time t, by -1, to get

$$\begin{aligned} \frac{dx}{dt} &= -x + f_1(x,y) \\ \frac{dy}{dt} &= -y + f_2(x,y) \end{aligned} \tag{3}$$

where

$$f_1(x,y) = -c_{12}\tanh(y), \quad f_2(x,y) = -c_{21}\tanh(x)$$

We define the nullcline sets

$$\mathrm{N}_x = \{-x + f_1(x,y) = 0\} \quad \text{and} \quad \mathrm{N}_y = \{-y + f_2(x,y) = 0\}. \tag{4}$$

and also define the "pockets" i.e.,

$$\mathrm{P}_{++} = \{f_1(x,y) \geq 0,\ \&\ f_2(x,y) \geq 0\},\ \mathrm{P}_{+-} = \{f_1(x,y) \geq 0,\ \&\ f_2(x,y) \leq 0\},$$

$$\mathrm{P}_{-+} = \{f_1(x,y) \leq 0,\ \&\ f_2(x,y) \geq 0\},\ \mathrm{P}_{--} = \{f_1(x,y) \leq 0,\ \&\ f_2(x,y) \leq 0\}.$$

The pockets are closed regions where the direction of the flow is trivial in the sense that it is oriented only toward a unique quadrant. The nullcline curves and their pockets are shown in Figure 12 in the case where c_{12} and c_{21} are negative and satisfy $c_{12}c_{21} > 1$.

In general limit cycles are quite difficult to detect. However, we can state the following theorem.

Theorem 1 *The system of equations (1) does not admit any limit cycle.*

POOF OF THEOREM 1: we distinguish two cases: either $0 < c_{12}c_{12} < 1$ and $c_{12}c_{12} > 1$, or $c_{12}c_{21} < 0$.

Case 1: $0 < c_{12}c_{12} < 1$ and $c_{12}c_{12} > 1$. We first state the following lemma.

Lemma 1 *Let Y be a 2-dimensional vector field and Γ be a limit cycle of Y. Then Γ intersects the interior of each pocket.*

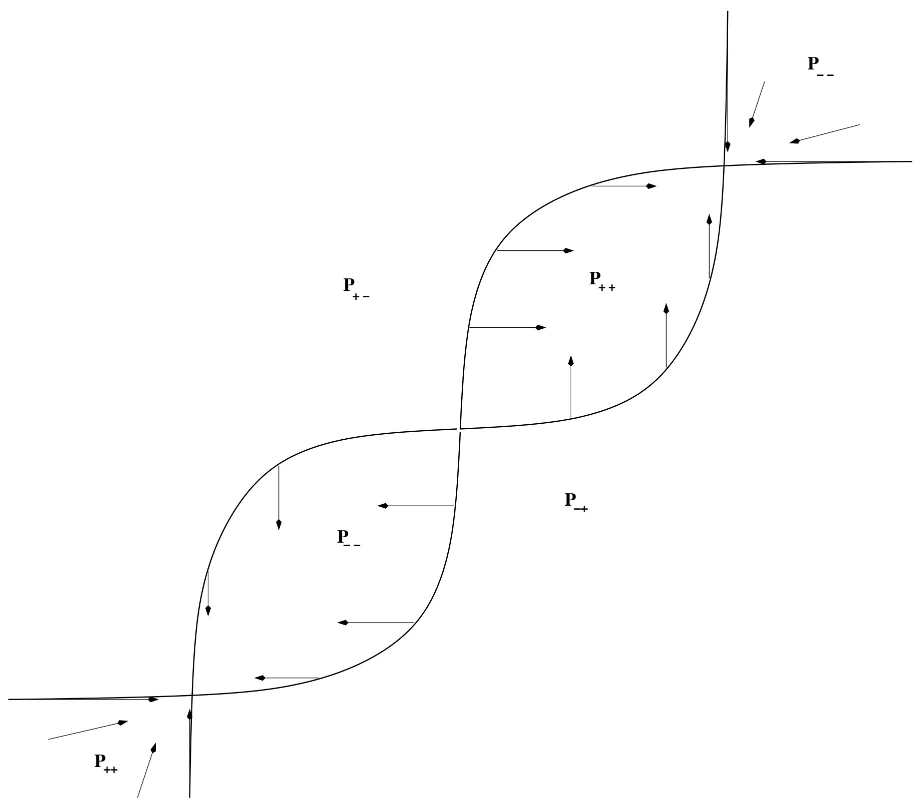

Figure 12: In the case where the two curves intersect at three points, i.e., $c_{12}c_{21} > 1$ the origin is a saddle but the other points are sinks. The figure consists of the union of 4 pockets P_{++}, P_{--} (both consisting of 2 connected components) P_{+-} and P_{-+} (being connected sets). We can deduce the flow at the boundary of the pockets i.e., on the nullcine. Observe that for any point located at the boundary δP_{++} (or δP_{--}) of P_{++} (respectively P_{--}) the flow enters P_{++} (respectively P_{--}), implying both sets to be invariant under the flow. Observe that from Figure 13, in the case where $c_{12}c_{21} \leq 1$, P_{++} and P_{--} possess only one connected component, but both sets are still invariant under the flow.

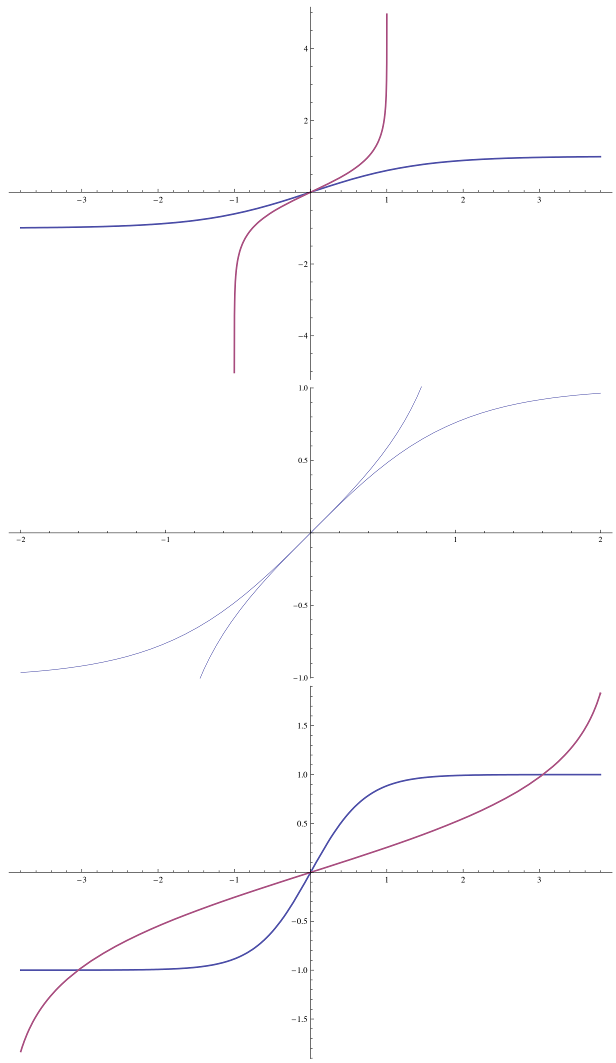

Figure 13: We distinguish the following configurations. Either $0 < c_1c_2 < 1$, $c_2c_1 > 1$ or $c_1c_2 < 0$. In the first configuration, we assume that $c_1 > 0,\ c_2 > 0$. When $1 < c_1c_2$ the nullcline sets intersect one another only once at the origin which is a sink. However, as the product c_1c_2 increases and crosses the value 1, a pitchfork bifurcation occurs. The origin is no longer a sink but a source, and the two sinks bifurcate into Q_+ and Q_- respectively located in the first quadrant and in the third quadrant. In the case where c_{12} and c_{21} are of opposite sign, the nullclines intersect only once at the singularity.

Proof: Let Γ be a limit cycle of Y. Since Γ is compact there exists a (minimal) rectangle

$$\mathrm{B} = [x_{\min}, x_{\max}] \times [y_{\min}, y_{\max}]$$

such that $\Gamma \subset \mathrm{B}$ and here exists 4 points P_N, P_S, P_W and P_E belonging to Γ where the boundary $\delta\mathrm{B}$ of B is tangent to Γ. Denote by $T_N,\ T_S,\ T_W,\ T_E$ the corresponding values of the time t such that

$$\Gamma(T_N) = P_N,\ \Gamma(T_S) = P_S,\ \Gamma(T_W) = P_W,\ \text{and } \Gamma(T_E) = P_E.$$

This latter means that

$$Y(P_N) = \dot{\Gamma}(T_N),\ Y(P_S) = \dot{\Gamma}(T_S),\ Y(P_W) = \dot{\Gamma}(T_W), Y(P_E) = \dot{\Gamma}(T_E).$$

and

$$\dot{y}(T_N) = 0,\ \text{and } \dot{x}(T_N) \neq 0.$$

Therefore for $t \sim T_N$, $\dot{x}(t) \neq 0$ by continuity. Without the loss of generality we can assume Γ to be counterclockwise oriented i.e., $\dot{x}(t) < 0$. Moreover, $\dot{y}(t)$ changes its sign at $t = T_N$. Therefore Γ crosses the interior of 2 pockets: P_{-+} and P_{--}.

Revisiting this argument at $t = t_W$ and $t = t_S$ leads to $\dot{x}(T_W) = 0,\ \dot{y}(T_S) = 0$ and $\dot{y}(T_N) < 0,\ \dot{x}(T_S) > 0$. Therefore for $t \sim T_W$, $\dot{y}(t) < 0$ and for $t \sim T_S$, $\dot{x}(t) > 0$. Moreover, $\dot{x}(t)$ changes its sign at $t = T_W$ and $\dot{y}(t)$ changes its sign at $t = T_S$. Therefore Γ crosses the interior of P_{+-} and P_{++}. □

Assume that eqs. (1) admits a limit cycle Γ. From Lemma 1, Γ should intersect the interior of P_{++}. However, both P_{++} and P_{--} are invariant under the flow. This means that once an orbit enters in one domain or the other, it never leaves that domain, which is impossible for a limit cycle. □

Case 2: $c_{12}c_{12} < 0$. We now write eqs. (1) as

$$\mathrm{X} : \left\{ \begin{array}{lcl} \dot{x} & = & -x + a\tanh(y) \\ \dot{y} & = & -y - b\tanh(x) \end{array} \right. \tag{5}$$

where X denotes the vector field associated with (5) and where, by assumption, $a = c_{12} \geq 0$ and $b = -c_{12} \geq 0$. Define the following map

$$H : \mathbb{R}^2 \to \mathbb{R},\ (x, y) \mapsto b\int_0^x \tanh(u)du + a\int_0^y \tanh(u)du.$$

From (5) it follows that

$$\begin{aligned} \mathrm{X}(H) = \dot{H} &= b\tanh(x)(-x + a\tanh(y)) + a\tanh(y)(-y - b\tanh(x)) \\ &= -bx\tanh(x) - ay\tanh(y). \end{aligned} \tag{6}$$

It follows that $X(H) = 0$ if and only if $(x, y) = (0, 0)$ otherwise $X(H) < 0$ and therefore H is a Lyapunov function and no limit cycles are allowed, ending the proof of Theorem 1. □

3.7 More than Two Actors - Numerical Results

The dynamical structure of this system with more than two actors is very complex and beyond the scope of the present paper. However, we do report here the results of some preliminary numerical studies in the regime of 3 to 10 actors and of analytical studies on the existence of limit cycles. If all the actors respond with negative feedback, the network will spontaneously break into two groups. The actors in each group all share the same emotional state (positive or negative), and the two groups will have opposite emotional states. We have also found (both numerical and analytically) that for this case with strong feedback the neutral point at $(0, 0, ..., 0)$ is a higher dimension unstable saddle. If the set of actors have mixed positive and negative feedback the network may display transient nonlinear oscillations, as illustrated in Figure 14. We are now beginning to determine the dynamics present in the different regions of the parameter space.

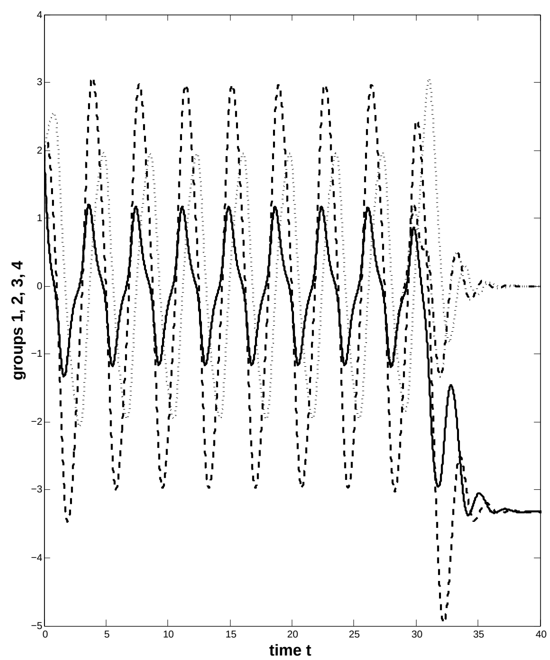

Figure 14: Numerical integration of four groups with mixed positive and negative feedback interactions illustrating transient oscillations. The feedback constants c_{ij} from group i to group j are $c_{11} = 0$, $c_{21} = 3$, $c_{31} = -3$, $c_{41} = -3$, $c_{12} = 3$, $c_{22} = 0$, $c_{32} = -3$, $c_{42} = 3$, $c_{13} = -3$, $c_{23} = 3$, $c_{33} = 0$, $c_{43} = 3$, $c_{14} = -3$, $c_{24} = 3$, $c_{34} = -3$, $c_{44} = 0$ and the initial condition for each group is $x_i = 2$.

3.8 More than Two Actors - Analytical Methods to Detect Limit Cycles

In higher dimension, the study of equilibria and their stability follows the same strategy as in dimension 2: the (Lyapounov) stability of each equilibrium is guaranteed when the eigenvalues of the linearization matrix of the vector field in eqs. (1) at the corresponding equilibrium have a negative real part. Singularities and their stability can be studied with methods from linear algebra. In dimension 2, a complete make up of the phase portrait can be deduced after knowing the loci of the limit cycles.

However, although this latter property is easy to state, since integrating a flow analytically is in most cases impossible, limit cycles are in general hard to detect, and the higher the dimension is, the more delicate is the task. At the mathematical level, tools to detect limit cycles are rather poor: for instance, thanks to Poincaré Bendixon Theorem, the existence of a limit cycle is deduced when the flow of the corresponding vector field enters an annulus and never escapes. This argument does not work in higher dimension and most of the time, the detection of limit cycles and more complicated dynamics is difficult. Indeed, unlike in the 2-dimensional, knowing the loci of limit cycles is not enough in order to understand the topology of the phase portrait. For example, even a 3-dimensional vector-field can exhibit very complicated dynamics such as suspended horseshoes and strange attractors.

The category of the system we consider here well illustrates this thought. We present here a family of systems that do not have limit cycles. Consider the following system

$$Z : \left\{ \begin{array}{rcl} \dot{x} & = & -x + \alpha_{12}\tanh(y) + \alpha_{13}\tanh(z) \\ \dot{y} & = & -y - \alpha_{21}\tanh(x) + \alpha_{23}\tanh(z) \\ \dot{z} & = & -z - \alpha_{31}\tanh(x) + \alpha_{32}\tanh(y). \end{array} \right. \tag{7}$$

Assume moreover that the above constants $\alpha_{21},\ \alpha_{12},\ \alpha_{31},\ \alpha_{13},\ \alpha_{23},\ \alpha_{32}$ are non-negative and

$$\det \begin{vmatrix} 0 & \alpha_{12} & \alpha_{13} \\ -\alpha_{21} & 0 & \alpha_{23} \\ -\alpha_{31} & -\alpha_{32} & 0 \end{vmatrix} = 0 \tag{8}$$

Then system (7) does not have a limit cycle. To show the non-existence of limit cycles in this case, we show that there exists non-negative constants A, B, C such that the function

$$(x, y, z) \mapsto H(x, y, z) = A\log(\cosh(x)) + B\log(\cosh(y)) + C\log(\cosh(z))$$

is a Lyapounov function.

A straightforward computation shows that the time derivative of the Lyapunov function along the vector field is given by

$$\begin{aligned}\dot{H} &= -Ax\tanh(x) - By\tanh(y) - Cz\tanh(z) \\ &+ (A\alpha_{12} - B\alpha_{21})\tanh(x)\tanh(y) \\ &+ (A\alpha_{13} - C\alpha_{31})\tanh(x)\tanh(z) + (B\alpha_{23} - C\alpha_{32})\tanh(y)\tanh(z)\end{aligned} \tag{9}$$

Observe that condition (8) implies that

$$\alpha_{21}\alpha_{13}\alpha_{32} = \alpha_{12}\alpha_{31}\alpha_{23} \tag{10}$$

If all coefficients are zero, the system is linear and contracting and therefore there are no limit cycles. In the case where not all coefficients are zero, without lost of generality we can assume $\alpha_{31} > 0$. In this case we are looking a tuple (A, B, C) solution of the equation

$$\left\{\begin{aligned} A\alpha_{12} - B\alpha_{21} &= 0 \\ A\alpha_{13} - C\alpha_{31} &= 0 \\ B\alpha_{23} - C\alpha_{32} &= 0 \end{aligned}\right. \tag{11}$$

We first set $A = \alpha_{21}$. From the first line of (11) we get

$$B = \alpha_{12} \tag{12}$$

and together with the 2nd line of (11)

$$\alpha_{13}\alpha_{21}/\alpha_{31} = C. \tag{13}$$

Observe that since (10) holds, (12) and (13) do not contradict the 3rd line of (11). With above choice of A ,B and C, with (11) it follows that $\dot{H} \leq 0$ and $\dot{H} = 0$ if and only if $(x, y, z) = 0$. This means that H is a Lyapounov function and therefore Z does not admit any limit cycle.

4 What is Learned from the Model and its Analysis

The validity of this model is supported by the fact that some of the results of the model are consistent with previous intuitions, observations, and experimental findings. The model is also valuable in that it makes new and testable predictions.

4.1 Results from the Model that Match Previous Observations and Experiments

First, the primary results of the model, that both actors reach similar emotional states in the case of positive-positive feedback and opposite emotional states in negative-negative feedback, is strongly supported by extensive observations of conflict resolution scholars and practitioners in conflicts characterized by cooperation or competition between the actors [5,6]. Second, the change in dynamics above a threshold of the strength of the feedback between the actors present in the model has been noted in both observational data and other theoretical models of conflicts [17]. Third, the sensitivity of the dynamics on parameters and initial conditions in the model has been observed in international relations [2] and in economics from both macroeconomics and financial data [8,19]. Fourth, the prediction of the model that an intractable negative-negative feedback conflict can be switched into a neutral state is supported by the work developed by Osgood in the 1960s on the gradual reduction in tension (GRIT) strategy [20].

4.2 Predictions of the Model

A strength of any mathematical model is the ability to make clear, testable, predictions that can later be confirmed or contradicted. Here, we have used both numerical and analytical methods to derive the novel dynamical properties of this model, and how they depend on the parameters. We now summarize those predictions in this section and in the next section illustrate how they can be tested by social psychological laboratory experiments. The numerical and analytical analysis of our models predicts the following:

1) A conflict with positive-negative feedback (cooperation-competition) between the actors should produce transient oscillations in the emotional states of the actors such as those illustrated in Figure 9. The period of those transient oscillations should be inversely related to the strength of the feedback between the actors. Both actors should evolve toward a neutrally stable state, rather than a clear winner and clear loser.

2) If the parameters that define the two actors in a conflict remain constant, then there cannot be sustained continuous oscillations of the states of the actors. That is, the dynamics of this system does not admit the existence of a limit cycle. For systems with three or more actors we have shown analytically that there are some conditions on the parameters under which there are no limit cycles. We cannot prove the existence (or non-existence) of limit cycles when those conditions are not met. Numerically, we have found conditions under which there are transient oscillations, but we have not found conditions under which there are stable limit cycles.

3) There is an advantage to be gained in a conflict if one actor alone can transiently switch their feedback. Such a strategy has been previously noted by Gottman et al. [15,16] in the "repair" terms of their model of the interactions of married couples. What is new here in the analysis of our model is that we predict that the frequency of these events should be inversely proportional to the inertia of the actors and that the duration of these transient switches should be inversely proportional to the strength of the feedback between the two actors.

4.3 Design of Experiments to Test the Predictions of the Model

Collecting and interpreting data from observations and performing these experiments is far beyond the scope of this theoretical paper which is focused on defining this model and determining its dynamical characteristics. However, we note briefly here how these predictions could in principle be tested.

Predictions 1 and 2 above, on the presence of transient, but not sustained oscillations, for constant parameters in a conflict can be tested by observation of the emotional state of individuals, as done by Gottman et al. [15,16], by coding facial expressions and utterances or between groups in conflicts by the frequency of actions (e.g. deaths or bombings) or of words of different intensity used in newspaper reports.

Perhaps the most interesting prediction of this model is Prediction 3 that an actor can unilaterally swap loser and winner roles in a conflict by judicious timing of switching the nature of their feedback. Even more concretely, we predict that frequency of these events should be inversely proportional to the inertia of the actors and that the duration of these periods should be inversely proportional to the strength of the feedback between the two actors. These predictions can be tested with social psychology laboratory experiments in the following way. First, subjects can be given a survey test to identify "hot topic" and "cool topic" items for each subject. Then, these subjects can be paired with a confederate. The subject and confederate could alternate presenting statements and responding to each other's statements. For example, the confederate can be instructed to present always negative feedback responses. Most likely, this would induce the subject to start off by presenting negative responses to the confederate's statements. The model predicts that even though the confederate always responds in the same way, that the subject will switch between periods of negative and positive responses. The model also predicts that the duration of the period of positive responses will be shorter for the "hot topic" items than for the "cool topic" items.

5 Summary

Our ability to formulate and analyze this detailed mathematical model of the conflict between two actors has allowed us to determine the dynamical effects and steady states that arise from their cooperative and/or competitive behavior. The results of even this simple, nonlinear model has properties that are consistent with the practical experience of conflict resolution scholars and practitioners and experimental findings by social psychologists. It also provides new insights and testable predictions about conflicts.

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Appendix:

Gottman, Murray, Swanson, Tyson, and Swanson
Model of the Mathematics of Marriage.

Let x_t be the emotional state of the wife and y_t the emotional state of the husband each at time t. The model of Gottman et al. [16], eq. (9.6) p. 134 is given by the difference equations:

$$\begin{aligned} x_{t+1} &= r_1 x_t + a_1 + I_{yx}(y_t) \\ y_{t+1} &= r_2 y_t + a_2 + I_{xy}(x_{t+1}) \end{aligned} \tag{14}$$

(15)

where r_1 and r_2 are the inertia to change, b_1 and b_2 are constants intrinsic to each person that determines their uninfluenced set points when they are alone in isolation, and $I_{yx}(y_t)$ is the influence function of y on x, and $I_{xy}(x_{t+1})$ the influence function of x on y. Note, that in isolation, over long times, the state of each person approaches their uninfluenced set point of x_s (uninfluenced) $= b_1/(1 - r_1)$ and y_s (uninfluenced) $= b_2/(1 - r_2)$ respectively. We study the differential form of these equations because doing so allows us to use standard software for numerical integration and because it makes available to us powerful theorems and analytical methods that are only applicable to continuous (and not discrete) systems so that we can more completely derive the dynamics. Perhaps this is also even more realistic as it represents a continuous interaction between the actors rather than alternating discrete equally timed events represented by the discrete model. We subtract x_t and y_t respectively from both equations, divide by $dt = 1$ time step to transform these difference equations into the differential equations:

$$\begin{aligned} \frac{dx}{dt} &= m_1 x + b_1 + f_1(y, x) \\ \frac{dy}{dt} &= m_2 x + b_2 + f_2(x, y) \end{aligned} \tag{16}$$

where $m_1 = r_1 - 1$, $m_2 = r_2 - 1$, $b_1 = a_1$, $b_2 = a_2$, $f_1(y, x) = I_{yx}(y)$ and $f_2(x, y) = I_{xy}(x_{t+1})$.

Let x and y be the state of each group at time t. In order to have small influences between the groups at small values of x and y, plateaus at large values of x and y, and to study the effects of positive and negative feedback between the groups we choose the influence functions:

$$\begin{aligned} f_1(y,x) &= c_1 \tanh(y) \\ f_2(x,y) &= c_2 \tanh(x) \end{aligned} \tag{17}$$

where $c_1 > 0$ corresponds to positive feedback from group y, $c_2 > 0$ corresponds to positive feedback from group x, $c_1 < 0$ corresponds to negative feedback from group y, and $c_2 < 0$ corresponds to negative feedback from group x. The equations that define the model are then:

$$\begin{aligned} \frac{dx}{dt} &= m_1 x + b_1 + c_1 \tanh(y) \\ \frac{dy}{dt} &= m_2 x + b_2 + c_2 \tanh(x) \end{aligned} \tag{18}$$

Critical Points

The "uninfluenced set points" the stable emotional state of each group in isolation are found by setting $dx/dt = dy/dt = 0$ and $c_1 = c_2 = 0$, in eq. 18 which yield:

$$\begin{aligned} x_s\ (uninfluenced) &= -\frac{b_1}{m_1} \\ y_s\ (uninfluenced) &= -\frac{b_2}{m_2} \end{aligned} \tag{19}$$

$$\tag{20}$$

The nullclines defined by $dx/dt = 0$ and $dy/dt = 0$ in eq. 18 are given by

$$x = \frac{-b_1 - c_1 \tanh(y)}{m_1}, \quad y = \frac{-b_2 - c_2 \tanh(x)}{m_2} \tag{21}$$

As Gottman et al. cleverly point out, the nullclines for x and y are the influence functions scaled (stretched or compressed) respectively by m_1 and m_2, and translated (moved up or down) respectively by b_1 and b_2.

Stability Analysis

We analyze the linear stability around a critical point $x = x_s$ and $y = y_s$ by substituting

$$\begin{aligned} x &= x_s + x' \\ y &= y_s + y' \end{aligned} \tag{22}$$

into eq. 18. Dropping the prime superscript, these equations become

$$\begin{aligned}\frac{dx_s}{dt} + \frac{dx}{dt} &= m_1 x_s + m_1 x + b_1 + c_1 \tanh(y_s + y) \\ \frac{dy_s}{dt} + \frac{dy}{dt} &= m_2 y_s + m_2 y + b_2 + c_2 \tanh(x_s + x)\end{aligned} \tag{23}$$

Using the Taylor series approximation

$$\begin{aligned}\tanh(y_s + y) &= \tanh(y_s) + y[\text{sech}^2(y_s)] \\ \tanh(x_s + x) &= \tanh(x_s) + x[\text{sech}^2(x_s)]\end{aligned} \tag{24}$$

and the fact that $dx/dt = dy/dt = 0$ at a critical point, namely

$$\begin{aligned}\frac{dx_s}{dt} &= m_1 x_s + b_1 + c_1 \tanh(y_s) = 0 \\ \frac{dy_s}{dt} &= m_2 y_s + b_2 + c_2 \tanh(x_s) = 0\end{aligned} \tag{25}$$

eq. 23 now becomes

$$\begin{aligned}\frac{dx}{dt} &= m_1 x + c_1 y \text{sech}^2(y_s) \\ \frac{dy}{dt} &= m_2 y + c_2 x \text{sech}^2(x_s)\end{aligned} \tag{26}$$

The eigenvalues of eq. (26) are then given by the solutions to

$$\begin{vmatrix} m_1 - \lambda & c_1 \text{sech}^2(y_s) \\ c_2 \text{sech}^2(x_s) & m_2 - \lambda \end{vmatrix} = 0 \tag{27}$$

which are

$$\lambda = (\frac{1}{2})\{(m_1 + m_2) \pm [(m_1 - m_2)^2] + 4c_1 c_2 \text{sech}^2(x_s)\text{sech}^2(y_s)]^{\frac{1}{2}}\} \tag{28}$$

For the models studied here

$$m_1 = m_2 = m \tag{29}$$

and thus eq. (28) simplifies to

$$\lambda = m \pm \sqrt{c_1 c_2 \text{sech}^2(x_s)\text{sech}^2(y_s)} \tag{30}$$

Case I: $|c| < |m|$.

The critical points in this regime are only at $x_s = y_s = 0$ and therefore eq. (30) becomes

$$\lambda = m \pm \sqrt{c_1 c_2} \tag{31}$$

For either positive feedback between both groups ($c_1 = c_2 > 0$) or negative feedback between both groups ($c_1 = c_2 < 0$)

$$c_1 c_2 \quad > \quad 0 \tag{32}$$

Since $m < 0$ and $|c| < |m|$, both eigenvalues $\lambda < 0$ and so $x_s = y_s = 0$ is stable. When there is positive feedback from one group and negative feedback from the other group, then

$$c_1 c_2 \quad < \quad 0 \tag{33}$$

and so the eigenvalues are complex, but with real part $m < 0$. Thus, this critical point is a spiral. The period of the oscillations around this spiral is then $2\pi/\sqrt{(c_1 c_2)}$.

Case II: $|c| > |m|$.

We first consider the critical point at $x_s = y_s = 0$. As in Case I, λ is given by eq. (31). For either positive feedback between both groups ($c_1 = c_2 > 0$) or negative feedback between both groups ($c_1 = c_2 < 0$) since $m < 0$ and $|c| > |m|$, there is one negative and one positive eigenvalue and thus this critical point is a saddle and unstable. When there is positive feedback from one group and negative feedback from the other group, both the eigenvalues are complex, but since the real part $m < 0$, this is a stable spiral.

We now consider the two other critical points which are present when there is either positive or negative feedback between both groups. These critical points occur for $x_s >> 0$ and $y_s >> 0$ and thus $\tanh(x_s) = \tanh(y_s) \sim 1$. Hence from eq. (21) with $b_1 = b_2 = 0$ the critical points are given by

$$x_s \quad = \quad -\frac{c_1}{m_1} \tag{34}$$

$$y_s \quad = \quad -\frac{c_2}{m_2} \tag{35}$$

Since

$$\mathrm{sech}^2(y_s) \quad = \quad [\frac{2}{e^{y_s} + e^{-y_s}}]^2 = 4e^{-2y_s} << 1$$

$$\mathrm{sech}^2(x_s) \quad = \quad [\frac{2}{e^{x_s} + e^{-x_s}}]^2 = 4e^{-2x_s} << 1 \tag{36}$$

$$\tag{37}$$

the eigenvalues from eq. (30) become

$$\lambda \quad = \quad m \pm \varepsilon \tag{38}$$

where $\varepsilon << 1$. Since $m < 0$, both eigenvalues are less than zero and both critical points are stable.

Stability of an Isolated Group with Negative or Positive feedback

As noted in the text when there is negative (but not positive) feedback between both groups and the parameters and initial conditions of both groups are identical, then contrary to the analysis above, the critical point at $x_s = y_s = 0$ is the only stable point. This can be understood by examining the stability of the equation with one variable, as now both equations are identical and so the values of x and y evolve strictly together. This equation, with $m_1 = m_2 = m$, $b_1 = b_2 = 0$, and $c_1 = c_2 = c$ is then given by

$$\frac{dx}{dt} = mx + c\tanh(x) \qquad (39)$$

For negative feedback where $c < 0$, since $m < 0$, eq. (39) shows that $dx/dt > 0$ for all $x < 0$, and $dx/dt < 0$ for all $x > 0$. Thus there is only one critical point at $x = 0$ which is stable. This differs from the stability analysis above for the non-identical equations for x and y in eq. (18) where $x = 0$ and $y = 0$ is stable only for $|c| < |m|$ and two additional stable critical points appear when $|c| > |m|$. On the other hand, the situation is different for positive feedback where $c > 0$. We analyze the linear stability around $x_s = 0$ by substituting

$$x = x_s + x' \qquad (40)$$

(and dropping the prime superscript) which yields

$$\frac{dx}{dt} = (m + c)x \qquad (41)$$

Since $m < 0$, eq. (41) is therefore stable only if $|c| < |m|$. When $|c| > |m|$, the signs of dx/dt from eq. (39) for increasing x are: $+, -, +, -$. Thus there are three critical points, identified by the change in sign. The central one at $x = 0$ is unstable, while the two others are stable. Thus, the stability of eq. (41) for positive feedback is analogous to that derived above for the non- identical equations for x and y in eq. (18).

References:

[1] Coleman, P. T. 2003. Characteristics of protracted intractable conflict: Toward the development of a metaframework- I. *Peace and Conflict: Journal of Peace Psychology*, **9**, 1-37.

[2] Jervis, R. 1997. *System Effects: Complexity in Political and Social Life.* Princeton: Princeton University Press.

[3] Lederach, J. P. 1997. *Building Peace: Sustainable Reconciliation in Divided Societies.* Washington DC: US Institute of Peace.

[4] Deutsch, M., Coleman, P. T., & Marcus, E. C. 2006. *The Handbook of Conflict Resolution: Theory and Practice.* San Francisco: John Wiley & Sons.

[5] Deutsch, M. 1973. *The Resolution of Conflict: Constructive and Destructive Processes.* New Haven: Yale University Press.

[6] Deutsch, M. 2006. Cooperation and competition. In M. Deutsch, P. T. Coleman, & E. C. Marcus (Eds.). *The Handbook of Conflict Resolution: Theory and Practice.* (pp. 23-42), San Francisco: John Wiley & Sons.

[7] Boulding, K. E. 1962. *Conflict and Defense.* New York: Harper.

[8] Pruitt, D. G. 1969. Stability and sudden change in interpersonal and international affairs. *Journal of Conflict Resolution*, **13**, 18-38.

[9] Pruitt, D. G. 2006. A graphical interpretation of escalation and de-escalation presented at Dynamics and Complexity of Intractable Conflicts, Kamimierz, Poland, Oct. 19-22, 2006.

[10] Vallacher, R. R. & Nowak, A. 2005. Dynamical social psychology: finding order in the flow of human experience. In A. W. Kruglanski & E. T. Higgins (Eds.), *Social Psychology: Handbook of Basic Principles.* New York: Guilford Publications.

[11] Coleman, P. T., Vallacher, R., Nowak, A., & Bui-Wrzosinska, L. 2006. Dynamical systems approach to intractable conflict. unpublished manuscript.

[12] Bui-Wrzosinska, L. 2005. The dynamics of conflict in a school setting. Unpublished Masters thesis, Warsaw School for Social Psychology.

[13] Richardson, L. F. 1960. *Arms and Insecurity.* Pittsburg: Boxwood Press.

[14] Richardson, L. F. 1960. *Statistics of Deadly Quarrels.* Pittsburg: Boxwood Press.

[15] Gottman, J., Swanson, C., & Swanson, K. 2002. A general systems the-

ory of marriage: Nonlinear difference equation modeling of marital interaction. *Personality and Social Psychology Review*, **4**, 326-340.

[16] Gottman, J. M., Murray, J. D., Swanson, C. C., Tyson, R., & Swanson, K. R. 2002. *The Mathematics of Marriage*. Cambridge: MIT Press.

[17] Yiu, K. T. W. & Cheung, S. O. 2005. A catastrophe model of construction conflict behavior. Building and Environment, **41**, 438-447.

[18] Peters, E. E. 2001. Fractal Market Analysis: Applying Chaos Theory to Investment and Economics. Hoboken, NJ, Wiley, John & Sons.

[19] Mandelbrot B. B. & Hudson R. L.. 2004. *The (Mis)Behavior Of Markets.* New York, Basic Books.

[20] Conflict Research Consortium, Guy Burgess and Heidi Burgess, Co-Directors. (1999). Online Training Program on Intractable Conflict (OTPIC), http://www.colorado.edu/conflict/peace/treatment/grit.htm.